Pergamon

Appl. Math. Lett. Vol. 10, No. 4, pp. 5–9, 1997

0893-9659/97 $17.00 + 0.00

PII: S0893-9659(97)00050-5

The WTC and ARS Painlevé Tests

R. SCHMITZ
Forschungszentrum der Deutschen Telekom AG
Am Kavalleriesand 3
D-64295 Darmstadt, Germany

(Received January 1997; accepted February 1997)

Communicated by B. Fuchssteiner

Abstract—The WTC and ARS tests are important tools in identifying nonlinear PDEs which are linearizable by the method of the inverse-scattering-transform. In this paper we give an exact formulation of these tests, and it is shown that the WTC test is "stronger" than the ARS test, i.e., every PDE which satisfies the WTC test also satisfies the ARS test.

Keywords—Nonlinear PDEs, Painlevé test, Inverse scattering transform.

1. INTRODUCTION

In 1980, Ablowitz, Ramani and Segur [1,2] conjectured that if a nonlinear PDE in $(1+1)$ independent variables is solvable via the method of the inverse scattering transform (see [3] for general information), all its group-theoretical reductions to ODEs have the Painlevé property, that is, all movable singularities of the solutions of the reductions are poles. This conjecture is called the *Painlevé conjecture* in the literature. They also gave an algorithm called *singular point analysis* in order to determine whether an ODE has this property or not. Their conjecture, interpreted as a necessary condition for integrability, is called *ARS test* in the sequel.

In order to circumvent the need to determine all group-theoretical reductions (which may not exist, anyway) of a given PDE, Weiss, Tabor and Carnevale [4] proposed in 1983 an alternative test, which allowed the testing of PDE directly. It consists of inserting a formal ansatz of the form

$$u(x,t) = \phi^p(x,t)\sum_{j=0}^{\infty} u_j(x,t)\phi^j(x,t) \tag{1.1}$$

into the PDE. We will call this test the *WTC test.*

In Section 2, we give an exact formulation of these two tests and discuss the connection between the ARS and WTC test and the Painlevé property for ODEs and PDEs, respectively. In Section 3, a theorem is proved, which states that a PDE which satisfies the WTC test also satisfies the ARS test.

This work is part of the author's doctoral dissertation at the Technical University of Braunschweig. The author gratefully acknowledges constant encouragement by J. Jaenicke and K. Hardenberg and many valuable discussions with F. Galas.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

2. PAINLEVÉ TESTS AND PAINLEVÉ PROPERTY

We start by recalling the Painlevé test and property for ordinary differential equations in the complex domain.

DEFINITION 2.1. *A nonlinear ODE in the complex domain*

$$f\left(z, w; \dots, w^{(n)}\right) = 0 \tag{2.1}$$

has the Painlevé property if and only if its general solution has as movable singularities (i.e., depending on integration constants) only poles.

It is clear that a necessary condition for the Painlevé property of an ODE is that an ansatz for the general solution in the form of a formal Laurent series is possible. The singular point analysis of Ablowitz, Ramani and Segur is based on this necessary condition, which is not sufficient, however, as is shown in [2] by a counterexample. Consider the nonlinear ODE

$$w''(z) = (w'(z))^2 \frac{2w(z) - 1}{w(z) + 1}. \tag{2.2}$$

One can show that a formal Laurent series as solution ansatz is possible, but the general solution of the ODE is $w(z) = \tan(\log(Az - B))$, with a movable essential singularity along $z = B/A$.

In order to formulate the ARS test, we first need to know what a group-theoretical reduction is.

DEFINITION 2.2. *Consider the n^{th} order PDE*

$$F(x, t; u, \dots, \partial^\alpha u, \dots) = 0, \tag{2.3}$$

where F is analytic in all its arguments and α a multi-index with $|\alpha| \le n$. If the ansatz

$$u = U\left(x, t, w, (z(x, t))\right) \tag{2.4}$$

leads to an ODE

$$f\left(z; w, w', \dots, w^{(n)}\right) = 0, \tag{2.5}$$

then $f =: F/G = 0$ is called a group-theoretical reduction of $F = 0$, where G is a regular, projectable analytic Lie-group of transformations (cf. [5]), which can be determined by considering $z = \eta(x, t)$ and $w = \zeta(x, t, u)$ as analytic invariants of G.

Note that we have adopted here the *direct method* of Clarkson and Kruskal [6], which is somewhat more general than the classical approach to group-theoretical reductions, which can be found in the book by Olver [5], since it can lead to more reductions than the classical method.

DEFINITION 2.3. *An n^{th} order PDE $F = 0$ satisfies the ARS test if and only if for all its group-theoretical reductions $F/G = 0$, a formal ansatz for the general solution*

$$w(z) = (z - z_0)^p \sum_{j=0}^{\infty} a_j (z - z_0)^j \tag{2.6}$$

is possible (perhaps after a transformation), where $z_0 \in \mathbb{C}$ is arbitrary, $-p \in \mathbb{N}$, and $n - 1$ of the coefficients a_j are arbitrary.

In Definition 2.3, we have summarized the original ARS conjecture (which was formulated only for *classical* reductions, however), and their singular-point-analysis. The partial proof of the ARS conjecture given by McLeod and Olver in [7] can be easily extended to the nonclassical reductions of Clarkson and Kruskal. The ARS test is meant only as a necessary condition for integrability. It cannot be extended to a sufficient condition, since Clarkson [8] has shown that the MBBM-equation

$$u_t + u_x + u^2 u_x + u_{xxt} = 0 \tag{2.7}$$

which is thought not to be integrable by inverse scattering, since the interaction of solitary waves for (7) is inelastic, as numerical results show (see [9] for details), satisfies the ARS test.

The WTC test was introduced by Weiss, Tabor and Carnevale [3] in order to test the given PDE directly, without having to calculate all group-theoretical reductions. In the WTC test, the solution of the PDE is assumed to be a meromorphic function in x and t and is as such represented locally as the quotient of two analytic functions (see [10]).

DEFINITION 2.4. *An n^{th} order PDE $F(x,t;u,\dots,\partial^\alpha u,\dots) = 0$ satisfies the WTC test if and only if the PDE allows (perhaps after a transformation) a formal ansatz for the general solution*

$$u(x,t) = \phi^p(x,t)\sum_{j=0}^{\infty} u_j(x,t)\phi^j(x,t), \tag{2.8}$$

where $-p \in \mathbb{N}$, $\phi : \mathbb{C}^2 \to \mathbb{C}$ is an arbitrary analytic function, and $n-1$ of the functions u_j are arbitrary.

REMARK 2.5. The possible transformations may have the form $v = Q(u, u_x, u_t, \dots)$ with Q analytic in its arguments (cf. [7]), or they are so-called *hodograph transformations* [11]. These hodograph transformations include the interchange of dependent and independent variables and serve to transform quasilinear PDEs such as the Harry-Dym equation

$$u_t + 2\left(u^{-1/2}\right)_{xxx} = 0 \tag{2.9}$$

into semilinear equations. In its quasilinear form, although integrable via inverse scattering [12], the HD equation does not pass the WTC test, but it passes the test in its semilinear form (see [11] for details).

Generalizing the Painlevé property for ODEs to the two-dimensional case is not quite straightforward, because simply assuming meromorphicity of the solution on $\mathbb{C}^2$ would be too rigid, since nonpolar singularities could be introduced into the solution by the initial conditions and then move along the characteristic hypersurfaces. In view of this, the Painlevé property for partial differential equations was defined as follows (cf. [13]).

DEFINITION 2.6. *Let $F = 0$ be a PDE, $S \subset \mathbb{C}^2$ be a noncharacteristic, analytic hypersurface. $F = 0$ has the Painlevé property for partial differential equations if every solution of $F = 0$, which is analytic on $\mathbb{C}^2 \backslash S$, is meromorphic on $\mathbb{C}^2$.*

The following theorem shows that there is a relationship between the WTC test and the Painlevé property for PDEs analogous to the relation between a formal Laurent series for the solution of an ODE and the Painlevé property for ODEs.

THEOREM 2.7. *If a PDE $F = 0$ has the Painlevé property for partial differential equations, it satisfies the WTC test, if one excludes in the test those ϕ, which describe a characteristic hypersurface $C = \{(x,t) \mid \phi(x,t) = 0\}$.*

PROOF. Let u be a solution of $F = 0$, which is analytic on $\mathbb{C}^2\backslash S$. Since S is noncharacteristic, u is meromorphic, and analytic functions $g, h : \mathbb{C}^2 \to \mathbb{C}$ exist, where $u = g/h$. Now let $x_0 \in \mathbb{C}_2$, where $h(x_0) = 0$. Since u is analytic on $\mathbb{C}^n\backslash S$, we have $x_0 \in S$, which means $\phi(x_0) = 0$. This shows

$$\exists m \in \mathbb{N}\ \exists f : \mathbb{C}^2 \to \mathbb{C} \text{ analytic} : \phi^m = fh.$$

We now have $h = \phi^m/f$ and $u = g/h = gf/\phi^m$. Set $y = \phi(x,t)$. Since $k := gf$ is analytic, we can expand k into a power series round $y = 0$:

$$k(x) = \sum_{j=0}^{\infty} u_j(x,t)y^j,$$

where the u_j are analytic. Since $u = k/F^m$, the claim follows.

Clarkson, in [14], has constructed a counterexample which shows that the converse of the theorem is not true, again in anology to the one-dimensional case. Consider the nonlinear PDE

$$u_t^2 = 2uu_x^2 - \left(1 + u^2\right) u_{xx} \tag{2.10}$$

which satisfies the WTC test. Looking at the similarity reduction, $w(z) = u(x-t)$ yields exactly (2.2), so we have $u(x,t) = \tan(\log(A(x-t) - B))$ as a special solution of (2.10), with essential singularities along the noncharacteristic hypersurface $S = \{(x,t) \mid x - t = B/A\}$.

3. THE CONNECTION BETWEEN ARS TEST AND WTC TEST

In [8], Clarkson showed not only that the MBBM equation satisfies the ARS test, but also that it does not satisfy the WTC test, since the condition $\phi_{tt} = 0$ has to be fulfilled by the singularity manifold $\{(x,t) \mid \phi(x,t) = 0\}$. This condition, however, is fulfilled by the only similarity variable of the MBBM equation, namely $z = \eta(x,t) = x - ct$. This motivates the idea to use the singularity function ϕ as a similarity variable, which can be used to prove the following theorem.

THEOREM 3.1. *If a nonlinear PDE $F = 0$ satisfies the WTC test, it also satisfies the ARS test.*

PROOF. Let

$$F/G\left(z, w, w', \dots, w^{(n)}\right) = 0$$

be a group-theoretical reduction of $F = 0$ with the solution $w = w(z)$. Let w have a singularity at $z = z_0$. We need to show that we can expand w in a formal Laurent series in a neighborhood of z_0. Since $z = z(x,t)$ analytic, we have $w = w(z(x,t))$ is singular on the hypersurface

$$\Gamma = \{(x,t) \mid z(x,t) = z_0\}.$$

According to Definition 2.2, $w(x,t) = w(x,t,u(x,t))$ holds, where w is analytic in (x,t,u) and u a solution of $F = 0$. Because of the analyticity of w, singularites of w can only come from singularities of u, so u has to be singular on Γ. Our assumption that $F = 0$ satisfies the WTC test now yields

$$u(x,t) = (z_0 - z(x,t))^p \sum_{j=0}^{\infty} u_j(x,t)(z_0 - z(x,t))^j.$$

But $w = w(x,t,u)$ is analytic in u, so we may expand w in a powerseries in a neighborhood around $u = 0$. After we have introduced the similarity variable $z = z(x,t)$ in this powerseries, we have already found the Laurent series for w, since $w(x,t) = w(x,t,u(x,t)) \stackrel{!}{=} w(z)$.

REMARK 3.2. Again, the converse of this theorem is not true, as the example of the MBBM equation shows. Note that (2.10) and its similarity reduction (2.2) do not provide a counterexample to Theorem 3.1, since the theorem deals with the WTC and ARS tests of Definitions 2.3 and 2.4 and not with the Painlevé properties of Definitions 2.1 and 2.6.

REFERENCES

1. M.J. Ablowitz, A. Ramani and H. Segur, Nonlinear evolution equations and ordinary differential equations of Painlevé type, *Lett. Nuovo Cimento* **23**, 333–338, (1978).
2. M.J. Ablowitz, A. Ramani and H. Segur, A connection between nonlinear evolution equations and ordinary differential equations of P-Type I, *J. Math. Phys.* **21**, 715–721, (1980).
3. M.J. Ablowitz and P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge, (1991).
4. J. Weiss, M. Tabor and G. Carnevale, The Painlevé property for partial differential equations, *J. Math. Phys.* **24**, 522–526, (1983).
5. P.J. Olver, *Applications of Lie-Groups to Differential Equations*, Springer, Berlin, (1986).
6. P.A. Clarkson and M.D Kruskal, New similarity reductions of the Boussinesq Equation, *J. Math. Phys.* **30**, 2201–2213, (1989).
7. J.B. Mcleod and P.J. Olver, The connection between partial differential equations soluble by inverse scattering and ordinary differential equations of Painlevé type, *SIAM J. Math. Anal.* **14**, 488–506, (1983).
8. P.A. Clarkson, New similarity reductions and Painlevé analysis for the symmetric regularised long wave and modified Benjamin-Bona-Mahoney equations, *J. Phys. A: Math. Gen.* **22**, 3821–3848, (1989).
9. V.G. Mankhankov, Dynamics of classical solitons, *Phys. Rep.* **35**, 1–128, (1987).
10. L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, North-Holland, Amsterdam, (1973).
11. P.A. Clarkson, A.S. Fokas and M.J. Ablowitz, Hodograph transformations of linearizable partial differential equations, *SIAM J. Math. Appl. Math.* **49**, 1188–1209, (1989).
12. M. Wadati, K. Konno and Y. Ichikawa, A generalization of the inverse scattering method, *J. Phys. Soc. Japan* **46**, 1965–1966, (1979).
13. R.S. Ward, The Painlevé property for the self-dual gauge-field equations, *Phys. Rev. Lett.* **102A**, 279–282, (1984).
14. P.A. Clarkson, The Painlevé property and a partial differential equation with an essential singularity, *Physics Letters A* **109** (1985), 205–208, (1985).